ON THE PRINCIPAL RICCI CURVATURES OF A RIEMANNIAN 3-MANIFOLD

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ABSTRACT. Milnor [5] has shown that three-dimensional Lie groups with left invariant Riemannian metrics furnish examples of 3-manifolds with principal Ricci curvatures of fixed signature — *except* for the signatures $(-,+,+)$, $(0,+,-)$, and $(0,+,+)$. We examine these three cases on a Riemannian 3-manifold, and prove the following. If the manifold is closed, then the signature $(-,+,+)$ is not globally possible if it is of the form $-\mu, f, f$, with μ a positive constant and f a smooth function that never takes the values $0, -\mu$ (hence this also applies to the signature $(-,-,-)$). In the scalar flat and complete case, the signature $(0,+,-)$ is not globally possible if the eigenvalues are constants and the zero eigenspace is spanned by a unit length vector field with geodesic flow; if the manifold is closed and this vector field is also divergence-free, then $(0,+,-)$ is not possible even if the nonzero eigenvalues are not constant. Finally, on a connected and complete Riemannian 3-manifold, if $(0,+,+)$ occurs globally and the two positive eigenvalues are equal, then the universal cover splits isometrically.

1. STATEMENT OF RESULTS

Let (M,g) be a Riemannian 3-manifold. Its *Ricci transformation* is the smooth bundle endomorphism $\widehat{\mathrm{Ric}}\colon TM \longrightarrow TM$ whose fiberwise action $v \mapsto \widehat{\mathrm{Ric}}_p(v)$ is the unique vector satisfying $g(\widehat{\mathrm{Ric}}_p(v), w) = \mathrm{Ric}_p(v,w)$ for all $w \in T_pM$. $\widehat{\mathrm{Ric}}$ is self-adjoint with respect to g, with trace equal to the scalar curvature; furthermore, because the Weyl tensor vanishes in three dimensions, the curvature is determined by $\widehat{\mathrm{Ric}}$. The *principal Ricci curvatures* are the eigenvalues of the Ricci transformation. In the well known work [5], Milnor showed that three-dimensional Lie groups with left invariant Riemannian metrics furnish examples of manifolds with principal Ricci curvatures of globally fixed signature — *except* for the signatures $(-,+,+)$, $(0,+,-)$, and $(0,+,+)$. We examine these three exceptional cases here; our interest is global, since in fact any signature is *locally* possible on a 3-manifold [3]. Our first result gives a global obstruction to the signature $(-,+,+)$, and also, in fact, to the signature $(-,-,-)$. It generalizes a result in [13], which proved the case when f below is constant:

Theorem 1. *On a closed 3-manifold M, there is no Riemannian metric with principal Ricci curvatures $-\mu, f, f$ when μ is a positive number and f is a smooth function on M that never takes the values $0, -\mu$.*

Next, we examine the signature $(0,+,-)$, in particular when the manifold is scalar flat, which is to say, when the nonzero principal Ricci curvatures have the same magnitude: $0, f, -f$. Suppose in addition that the kernel of $\widehat{\text{Ric}}$ is spanned by a unit length vector field $\boldsymbol{k} \in \mathfrak{X}(M)$ with geodesic flow. This, and the Weinstein conjecture in dimension three [12], gives certain global obstructions to the signature $(0,+,-)$ in the scalar flat case:

Theorem 2. *Let (M,g) be a scalar flat Riemannian 3-manifold with principal Ricci curvatures of signature $(0,+,-)$. Let $\boldsymbol{k} \in \mathfrak{X}(M)$ be a unit length vector field in the zero eigenspace. If $\boldsymbol{k}$ has geodesic flow and (M,g) is complete, then the nonzero eigenvalues are not constant. If $\boldsymbol{k}$ has geodesic and divergence-free flow and M is closed, then $(0,+,-)$ is not possible.*

Finally, we examine the signature $(0,+,+)$, once again the case when the positive eigenvalues are equal. If $\boldsymbol{k} \in \mathfrak{X}(M)$ is nowhere vanishing and spans the zero eigenspace, then the case $0, f, f$ is equivalent to $R(\boldsymbol{k},\cdot,\cdot,\cdot) = 0$, where R is the Riemann 4-tensor; in particular, a 2-plane has zero sectional curvature if and only if it contains $\boldsymbol{k}$. Here we show that if the case $0, f, f$ occurs, then the universal cover must split isometrically:

Theorem 3. *Let (M,g) be a connected, complete Riemannian 3-manifold with positive scalar curvature. Let $\boldsymbol{k} \in \mathfrak{X}(M)$ be a unit length vector field along whose flow the scalar curvature is constant. If $R(\boldsymbol{k},\cdot,\cdot,\cdot) = 0$, then the universal cover of (M,g) splits isometrically as $\mathbb{R} \times \widetilde{N}$.*

In our proof of Theorem 3, positivity of the scalar curvature is key; in particular, this avoids the examples to be found in [10], which have constant negative scalar curvature and which do not split as in Theorem 3. The original version of our Theorem required in addition that S be bounded away from zero, and we kindly thank Benjamin Schmidt for informing us that Theorem 3 remains true without this assumption (the proof here is our own), and that in fact Theorem 3 can also be derived from results obtained in [9]. Indeed, as was kindly communicated to us by Wolfgang Ziller, Theorem 3 also follows from results obtained in [11] (which is valid in n dimensions).

2. Formalism and Conventions

The machinery we use is the *Newman-Penrose formalism* [6] (see also [7, Chapter 5]), which was adapted to Riemannian 3-manifolds in [1]. We use the notation "$\langle\,,\rangle$" to denote the metric g, and our sign convention for the Riemann tensor is

$$R(X,Y)Z \;=\; \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Given a local orthonormal frame in (M,g) of the form $\{\boldsymbol{k},\boldsymbol{x},\boldsymbol{y}\}$, begin by combining $\boldsymbol{x}$ and $\boldsymbol{y}$ into complex-valued vector fields

$$\boldsymbol{m} \;:=\; \frac{1}{\sqrt{2}}(\boldsymbol{x} - i\boldsymbol{y}) \quad , \quad \overline{\boldsymbol{m}} \;:=\; \frac{1}{\sqrt{2}}(\boldsymbol{x} + i\boldsymbol{y}), \tag{1}$$

and work with the complex triad $\{\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$ in place of $\{\boldsymbol{k},\boldsymbol{x},\boldsymbol{y}\}$ (doing this is not necessary, but it allows us to call upon equations already derived in [1]). Observe that $\langle \boldsymbol{m},\boldsymbol{m}\rangle = \langle \overline{\boldsymbol{m}},\overline{\boldsymbol{m}}\rangle = \langle \boldsymbol{k},\boldsymbol{m}\rangle = \langle \boldsymbol{k},\boldsymbol{m}\rangle = 0$, where, e.g., $\langle \boldsymbol{k},\boldsymbol{m}\rangle = \frac{1}{\sqrt{2}}(\langle \boldsymbol{k},\boldsymbol{x}\rangle - i\langle \boldsymbol{k},\boldsymbol{y}\rangle)$, while $\langle \boldsymbol{m},\overline{\boldsymbol{m}}\rangle = \langle \boldsymbol{k},\boldsymbol{k}\rangle = 1$. Since we will need the components of both the Riemann 4-tensor and Ricci tensor with respect to a complex triad $\{\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$, we observe here that the latter is given by

$$\mathrm{Ric}(\cdot,\cdot) \;=\; R(\boldsymbol{k},\cdot,\cdot,\boldsymbol{k}) + R(\boldsymbol{m},\cdot,\cdot,\overline{\boldsymbol{m}}) + R(\overline{\boldsymbol{m}},\cdot,\cdot,\boldsymbol{m}).$$

Next, define the following complex-valued quantities, which comprise the objects of study in the Newman-Penrose formalism:

$$\begin{aligned}\kappa \;&:=\; -\langle \nabla_{\boldsymbol{k}}\boldsymbol{k},\boldsymbol{m}\rangle \quad , \quad \rho \;:=\; -\langle \nabla_{\overline{\boldsymbol{m}}}\boldsymbol{k},\boldsymbol{m}\rangle \quad , \quad \sigma \;:=\; -\langle \nabla_{\boldsymbol{m}}\boldsymbol{k},\boldsymbol{m}\rangle,\\ &\varepsilon \;:=\; \langle \nabla_{\boldsymbol{k}}\boldsymbol{m},\overline{\boldsymbol{m}}\rangle \quad , \quad \beta \;:=\; \langle \nabla_{\boldsymbol{m}}\boldsymbol{m},\overline{\boldsymbol{m}}\rangle. \end{aligned} \tag{2}$$

These so-called *spin coefficients* were first introduced for null vector fields $\boldsymbol{k}$ on Lorentzian 4-manifolds in [6]. Observe that all the spin coefficients save for β can be defined solely along an integral curve of $\boldsymbol{k}$, e.g., by parallel translating two orthonormal vectors in $\boldsymbol{k}^{\perp}$. In any case, the first three spin coefficients in (2) are particularly important, as they encode geometric information regarding the flow of $\boldsymbol{k}$. For one, the flow of $\boldsymbol{k}$ is geodesic, by which is meant that $\nabla_{\boldsymbol{k}}\boldsymbol{k} = 0$, if and only if $\kappa = 0$ (if $\kappa = \varepsilon = 0$, then the vector fields $\boldsymbol{x}$ and $\boldsymbol{y}$ are parallel along the geodesic flow of $\boldsymbol{k}$). Next, denoting the divergence of $\boldsymbol{k}$ by $\operatorname{div}\boldsymbol{k}$, the real and imaginary parts of the spin coefficient ρ are given by

$$-\,2\rho \;=\; \operatorname{div}\boldsymbol{k} + i\,\omega, \tag{3}$$

where the smooth function $\omega := \langle \nabla_{\boldsymbol{y}}\boldsymbol{k},\boldsymbol{x}\rangle - \langle \nabla_{\boldsymbol{x}}\boldsymbol{k},\boldsymbol{y}\rangle$ vanishes if and only if the normal subbundle $\boldsymbol{k}^{\perp} \subset TM$ is integrable (this follows from Frobenius's theorem; note also that ω measures the "twist" of the flow of $\boldsymbol{k}$). Finally, the spin coefficient

$$\sigma \;=\; \frac{1}{2}\Big(\langle \nabla_{\boldsymbol{y}}\boldsymbol{k},\boldsymbol{y}\rangle - \langle \nabla_{\boldsymbol{x}}\boldsymbol{k},\boldsymbol{x}\rangle\Big) \;+\; \frac{i}{2}\Big(\langle \nabla_{\boldsymbol{y}}\boldsymbol{k},\boldsymbol{x}\rangle + \langle \nabla_{\boldsymbol{x}}\boldsymbol{k},\boldsymbol{y}\rangle\Big) \tag{4}$$

is the (complex) *shear* associated to $\boldsymbol{k}$'s flow: its magnitude $|\sigma|$ at any point determines whether an infinitesimal cross section of the flow deforms at that point into an ellipse of the same area. Regarding these three spin coefficients, it is straightforward to verify, for example, that $\boldsymbol{k}$ is parallel if and only if its flow is geodesic ($\kappa = 0$), shear-free ($\sigma = 0$), divergence-free ($\rho + \bar{\rho} = 0$), and twist-free ($\rho - \bar{\rho} = 0$). Certainly the vanishing of κ is independent of $\{\boldsymbol{x},\boldsymbol{y}\}$; so, too, is the vanishing of ρ and σ (see, e.g., [7, p. 327ff.]). Indeed, $|\sigma|^2$ is the determinant of the trace-free symmetric part of the left-hand matrix in (16) below, while ω^2 is the determinant of its skew-symmetric part. Therefore both $|\sigma|^2$ and ω^2 are frame-independent smooth functions on M; so, too, is ω, in the case when $\boldsymbol{x}$ and $\boldsymbol{y}$ are globally defined (indeed, the real and imaginary parts of all spin coefficients are smooth functions

of M in this case). Finally, note that the spin coefficient $\varepsilon = i\langle \nabla_{\boldsymbol{k}}\boldsymbol{x}, \boldsymbol{y}\rangle$ is purely imaginary in any frame.

Finally, let us rewrite, in terms of a local complex tetrad $\{\boldsymbol{k}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$, the components $R(\boldsymbol{k}, \overline{\boldsymbol{m}}, \boldsymbol{k}, \boldsymbol{m})$, $R(\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{k}, \boldsymbol{m})$, $R(\overline{\boldsymbol{m}}, \boldsymbol{m}, \boldsymbol{k}, \boldsymbol{m})$, $R(\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{m}, \overline{\boldsymbol{m}})$, and $R(\overline{\boldsymbol{m}}, \boldsymbol{m}, \boldsymbol{m}, \overline{\boldsymbol{m}})$ of the Riemann 4-tensor in terms of the spin coefficients $\kappa, \rho, \sigma, \varepsilon, \beta$. Though tediously derived, the resulting five curvature identities are, respectively,

$$\begin{aligned}
\boldsymbol{k}[\rho] - \overline{\boldsymbol{m}}[\kappa] &= |\kappa|^2 + |\sigma|^2 + \rho^2 + \kappa\bar{\beta} + \frac{1}{2}\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{k}), && (5)\\
\boldsymbol{k}[\sigma] - \boldsymbol{m}[\kappa] &= \kappa^2 + 2\sigma\varepsilon + \sigma(\rho + \bar{\rho}) - \kappa\beta + \mathrm{Ric}(\boldsymbol{m}, \boldsymbol{m}), && (6)\\
\boldsymbol{m}[\rho] - \overline{\boldsymbol{m}}[\sigma] &= 2\sigma\bar{\beta} + (\bar{\rho} - \rho)\kappa + \mathrm{Ric}(\boldsymbol{k}, \boldsymbol{m}), && (7)\\
\boldsymbol{k}[\beta] - \boldsymbol{m}[\varepsilon] &= \sigma(\bar{\kappa} - \bar{\beta}) + \kappa(\varepsilon - \bar{\rho}) + \beta(\varepsilon + \bar{\rho}) - \mathrm{Ric}(\boldsymbol{k}, \boldsymbol{m}) && (8)\\
\boldsymbol{m}[\bar{\beta}] + \overline{\boldsymbol{m}}[\beta] &= |\sigma|^2 - |\rho|^2 - 2|\beta|^2 + (\rho - \bar{\rho})\varepsilon - \mathrm{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}}) + \frac{1}{2}\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{k}). && (9)
\end{aligned}$$

We do the same with the two differential Bianchi identities

$$\begin{aligned}
(\nabla_{\boldsymbol{k}}R)(\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{m}, \overline{\boldsymbol{m}}) + (\nabla_{\boldsymbol{m}}R)(\boldsymbol{k}, \boldsymbol{m}, \overline{\boldsymbol{m}}, \boldsymbol{k}) + (\nabla_{\overline{\boldsymbol{m}}}R)(\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{k}, \boldsymbol{m}) &= 0,\\
(\nabla_{\boldsymbol{k}}R)(\overline{\boldsymbol{m}}, \boldsymbol{m}, \boldsymbol{m}, \overline{\boldsymbol{m}}) + (\nabla_{\boldsymbol{m}}R)(\overline{\boldsymbol{m}}, \boldsymbol{m}, \overline{\boldsymbol{m}}, \boldsymbol{k}) + (\nabla_{\overline{\boldsymbol{m}}}R)(\overline{\boldsymbol{m}}, \boldsymbol{m}, \boldsymbol{k}, \boldsymbol{m}) &= 0,
\end{aligned}$$

which then take the forms, respectively,

$$\begin{aligned}
&\boldsymbol{k}[\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{m})] \; - \; \frac{1}{2}\boldsymbol{m}[\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{k})] \; + \; \overline{\boldsymbol{m}}[\mathrm{Ric}(\boldsymbol{m}, \boldsymbol{m})] \; = \qquad (10)\\
&\qquad \kappa\,\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{k}) \; + \; \big(\varepsilon + 2\rho + \bar{\rho}\big)\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{m}) \; + \; \sigma\,\mathrm{Ric}(\boldsymbol{k}, \overline{\boldsymbol{m}})\\
&\qquad\qquad - \; \big(\bar{\kappa} + 2\bar{\beta}\big)\mathrm{Ric}(\boldsymbol{m}, \boldsymbol{m}) \; - \; \kappa\,\mathrm{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}})
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{m}[\mathrm{Ric}(\boldsymbol{k}, \overline{\boldsymbol{m}})] \; + \; \overline{\boldsymbol{m}}[\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{m})] \; - \; \boldsymbol{k}[\mathrm{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}}) - \frac{1}{2}\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{k})] \; = \qquad (11)\\
&\qquad (\rho + \bar{\rho})\big(\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{k}) - \mathrm{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}})\big) \; - \; \bar{\sigma}\mathrm{Ric}(\boldsymbol{m}, \boldsymbol{m}) \; - \; \sigma\mathrm{Ric}(\overline{\boldsymbol{m}}, \overline{\boldsymbol{m}})\\
&\qquad\qquad - \; \big(2\bar{\kappa} + \bar{\beta}\big)\mathrm{Ric}(\boldsymbol{k}, \boldsymbol{m}) \; - \; \big(2\kappa + \beta\big)\mathrm{Ric}(\boldsymbol{k}, \overline{\boldsymbol{m}}).
\end{aligned}$$

(For a fuller treatment of these derivations, consult [1].)

3. The signature $(-,+,+)$

Proof of Theorem 1. Let g be Riemannian metric on a closed 3-manifold M with globally constant principal Ricci curvatures $-\mu, f, f$, with μ a positive number and f a smooth function that never takes the values $0, -\mu$.

Consider first the case when M is simply connected. The smooth bundle endomorphism $\widehat{\mathrm{Ric}}+\mu I\colon TM \longrightarrow TM$ has nullity one at every point, hence its kernel $X := \ker(\widehat{\mathrm{Ric}}+\mu I)$ is a smooth real line bundle over M (see, e.g., [4, Theorem 10.34, p. 266]). As M is simply connected, X therefore has a smooth global section $\boldsymbol{k} \in \Gamma(X)$ of unit length. Since $\widehat{\mathrm{Ric}}$ is self-adjoint, it admits a local orthonormal basis of eigenvectors $\{\boldsymbol{k},\boldsymbol{u},\boldsymbol{v}\}$ such that

$$\widehat{\mathrm{Ric}}(\boldsymbol{u}) \;=\; f\boldsymbol{u} \quad , \quad \widehat{\mathrm{Ric}}(\boldsymbol{v}) \;=\; f\boldsymbol{v}. \tag{12}$$

It follows that with respect to the corresponding local complex tetrad $\{\boldsymbol{k},\boldsymbol{n},\overline{\boldsymbol{n}}\}$, where $\boldsymbol{n} := \frac{1}{\sqrt{2}}(\boldsymbol{u}-i\boldsymbol{v})$ (and whose corresponding spin coefficients below we denote with a subscript "$*$"), the components of the Ricci tensor satisfy

$$\mathrm{Ric}(\boldsymbol{k},\boldsymbol{n}) \;=\; \mathrm{Ric}(\boldsymbol{n},\boldsymbol{n}) \;=\; 0 \;\;,\;\; \mathrm{Ric}(\boldsymbol{k},\boldsymbol{k}) \;=\; -\mu \;\;,\;\; \mathrm{Ric}(\boldsymbol{n},\overline{\boldsymbol{n}}) \;=\; f. \tag{13}$$

Let us first assume that f satisfies $\boldsymbol{k}[f]=0$. Then, inserting (13) into the differential Bianchi identities (10) and (11) yields $\kappa_* \;=\; \rho_*+\bar{\rho}_*=0$, so that the flow of $\boldsymbol{k}$ is geodesic ($\kappa_*=0$) and divergence-free ($\rho_*+\bar{\rho}_*=0$). Inserting these into the curvature identity (5), its real part simplifies to

$$|\sigma_*|^2-\frac{\omega_*^2}{4} \;=\; \frac{\mu}{2}\cdot \tag{14}$$

Since $|\sigma_*|^2$ and ω_*^2 are frame-independent, (14) is frame-independent and holds at each point of M. We now replace the vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ with global ones more suited to the geometry, as follows. Consider the normal subbundle $\boldsymbol{k}^\perp \subset TM$ and the smooth bundle endomorphism

$$D\colon \boldsymbol{k}^\perp \longrightarrow \boldsymbol{k}^\perp \quad , \quad Z \;\mapsto\; D(Z) \;:=\; \nabla_Z\boldsymbol{k}. \tag{15}$$

(This is well-defined because $\boldsymbol{k}$ has unit length.) In terms of the spin coefficients ρ_* and σ_*, the matrix of D at a point p is given by

$$\begin{bmatrix} \langle \nabla_{\boldsymbol{u}_p}\boldsymbol{k},\boldsymbol{u}_p\rangle & \langle \nabla_{\boldsymbol{v}_p}\boldsymbol{k},\boldsymbol{u}_p\rangle \\ \langle \nabla_{\boldsymbol{u}_p}\boldsymbol{k},\boldsymbol{v}_p\rangle & \langle \nabla_{\boldsymbol{v}_p}\boldsymbol{k},\boldsymbol{v}_p\rangle \end{bmatrix} \;=\; \begin{bmatrix} -\mathrm{re}(\sigma_*) & \frac{\omega_*}{2}+\mathrm{im}(\sigma_*) \\ -\frac{\omega_*}{2}+\mathrm{im}(\sigma_*) & \mathrm{re}(\sigma_*) \end{bmatrix}\Bigg|_p\cdot \tag{16}$$

By virtue of (14), each D_p has two distinct eigenvalues $\pm\sqrt{\mu/2}$ (note the positivity of μ). Thus the smooth bundle endomorphisms $D \pm \sqrt{\mu/2}\,I\colon \boldsymbol{k}^\perp \longrightarrow \boldsymbol{k}^\perp$ have nullity one at every point, in which case $X_\pm \ker(D\pm\sqrt{\mu/2}\,I)$ admit nowhere vanishing global sections $\tilde{\boldsymbol{x}} \in \Gamma(X_-)$ and $\tilde{\boldsymbol{y}} \in \Gamma(X_+)$. Now replace these with the global vector fields

$$\boldsymbol{x} \;:=\; \tilde{\boldsymbol{x}} \quad , \quad \boldsymbol{y} \;:=\; -\langle \tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}\rangle\,\tilde{\boldsymbol{x}}+\tilde{\boldsymbol{y}}, \tag{17}$$

chosen to have unit length. Writing the matrix D with respect to this new global (orthonormal) frame $\{\boldsymbol{k},\boldsymbol{x},\boldsymbol{y}\}$, whose corresponding global complex tetrad we denote by $\{\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$, it follows from (16) and the identity $D\boldsymbol{x} = \sqrt{\mu/2}\,\boldsymbol{x}$ that its spin coefficient σ (not to be confused with σ_* above, though of course $|\sigma_*|^2 \;=\; |\sigma|^2$) satisfies $\mathrm{re}(\sigma) \;=\; -\sqrt{\mu/2}$ and $\mathrm{im}(\sigma) \;=\; \omega/2$. To

summarize, then, we have shown that there exists a global complex tetrad $\{\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$ whose corresponding spin coefficients κ, ρ, and σ satisfy

$$\kappa \;=\; 0 \quad , \quad \rho \;=\; -i\,\frac{\omega}{2} \quad , \quad \sigma \;=\; -\sqrt{\frac{\mu}{2}}+i\,\frac{\omega}{2}\cdot \tag{18}$$

(Being geodesic ($\kappa = 0$) and divergence-free ($\rho+\bar{\rho}=0$) is, of course, independent of the complex tetrad used.) The virtue of this particular complex tetrad is the form of its shear σ in (18), which nicely simplifies the curvature identities (5), (6), (7), and (9) above. Indeed, inserting (18) into the imaginary part of (5) yields $\boldsymbol{k}[\omega]=0$, which immediately implies that $\boldsymbol{k}[\sigma]=0$. This in turn simplifies (6) to $2\sigma\varepsilon=0$, from which we deduce that $\varepsilon=0$. Next, inserting (18) into (7) yields real and imaginary parts

$$\boldsymbol{x}[\omega] \;=\; 2\sqrt{\frac{\mu}{2}}\,\mathrm{div}\,\boldsymbol{y}-\omega\,\mathrm{div}\,\boldsymbol{x} \quad , \quad \sqrt{\frac{\mu}{2}}\,\mathrm{div}\,\boldsymbol{x}+\frac{\omega}{2}\,\mathrm{div}\,\boldsymbol{y} \;=\; 0, \tag{19}$$

where $\beta=\frac{1}{\sqrt{2}}\left(\langle\nabla_{\boldsymbol{y}}\boldsymbol{x},\boldsymbol{y}\rangle+i\langle\nabla_{\boldsymbol{x}}\boldsymbol{x},\boldsymbol{y}\rangle\right)=\frac{1}{\sqrt{2}}(\mathrm{div}\,\boldsymbol{x}-i\,\mathrm{div}\,\boldsymbol{y})$ (the latter because $\nabla_{\boldsymbol{k}}\boldsymbol{k}=0$). Finally, (9) simplifies via (14) and $\varepsilon=0$ to

$$\boldsymbol{x}[\mathrm{div}\,\boldsymbol{x}]+\boldsymbol{y}[\mathrm{div}\,\boldsymbol{y}] \;=\; -(\mathrm{div}\,\boldsymbol{x})^2-(\mathrm{div}\,\boldsymbol{y})^2-f,$$

which in turn further simplifies, via (19), to

$$\left(-\frac{\omega}{2\sqrt{\mu/2}}\,\boldsymbol{x}+\boldsymbol{y}\right)[\mathrm{div}\,\boldsymbol{y}] \;=\; -f. \tag{20}$$

But this is impossible on a closed manifold with f positive. This completes the proof in the case that M is simply connected. If M is not simply connected, then pass to its universal cover $\pi\colon(\widetilde{M},\tilde{g})\longrightarrow(M,g)$, which has principal Ricci curvatures $-\mu, f\circ\pi, f\circ\pi$. Repeating our argument on $(\widetilde{M},\tilde{g})$ with corresponding global orthonormal basis of eigenvectors $\{\boldsymbol{K},\boldsymbol{X},\boldsymbol{Y}\}$, we once again arrive at (20). Although $(\widetilde{M},\tilde{g})$ need not be compact, we still obtain a contradiction because $\mathrm{div}\,\boldsymbol{Y}$ must be bounded in $\widetilde{M}$. The reason is because $d\pi(\boldsymbol{Y})\in\mathfrak{X}(M)$ is equal to $\boldsymbol{y}$ in (17) up to sign; but as $|\mathrm{div}\,\boldsymbol{y}|$ is a continuous function on M, it is bounded (observe that $\boldsymbol{y}$ in (17), though in general defined only locally when M is not simply connected, is nonetheless unique up to sign). Hence $\mathrm{div}\,\boldsymbol{Y}$ is bounded in $\widetilde{M}$, which contradicts (20) on $\widetilde{M}$ (written with respect to $\{\boldsymbol{K},\boldsymbol{X},\boldsymbol{Y}\}$).

Now suppose that f is a smooth function on M that never takes on the values $0,-\mu$, but that it does not necessarily satisfy $\boldsymbol{k}[f]=0$. Recalling D in (15) and (16) above (with $\mathrm{div}\,\boldsymbol{k}$ reinstated), begin by defining the function

$$H \;:=\; \det D-\frac{\mu}{2} \;=\; \frac{\omega^2}{4}-|\sigma|^2+\frac{(\mathrm{div}\,\boldsymbol{k})^2}{4}-\frac{\mu}{2}\;. \tag{21}$$

Even though f is no longer assumed to be constant, observe that the first differential Bianchi identity (10) nonetheless yields $\kappa=0$, so that $\boldsymbol{k}$ still has geodesic flow (the second differential Bianchi identity (11) now yields

$\boldsymbol{k}[f] = -(\operatorname{div}\boldsymbol{k})(\mu + f)$, to which we will return later). With $\kappa = 0$, the real and imaginary parts of (5) are, respectively,

$$\boldsymbol{k}[\operatorname{div}\boldsymbol{k}] \;=\; 2H - (\operatorname{div}\boldsymbol{k})^2 + 2\mu \quad , \quad \boldsymbol{k}[\omega] \;=\; -(\operatorname{div}\boldsymbol{k})\,\omega, \tag{22}$$

while (6), via the identity $\varepsilon + \bar{\varepsilon} = 0$ and the fact that $\mathrm{Ric}(\boldsymbol{m},\boldsymbol{m}) = 0$ in a local complex tetrad satisfying (12) (with f in place of λ), implies $\boldsymbol{k}[|\sigma|^2] = -2(\operatorname{div}\boldsymbol{k})\,|\sigma|^2$. (Since $|\sigma|^2$ is frame-independent, so is this equation.) These three equations combine to yield the following evolution equation for H along the flow of $\boldsymbol{k}$:

$$\boldsymbol{k}[H] \;=\; -(\operatorname{div}\boldsymbol{k})\,H. \tag{23}$$

This equation implies that along any integral curve $\gamma(s)$ of $\boldsymbol{k}$, either $H\circ\gamma$ is nowhere zero or else it vanishes identically. We claim that H cannot vanish; for if it does, inserting $\theta(s) := (\operatorname{div}\boldsymbol{k}\circ\gamma)(s)$ into equation (22) yields

$$\frac{d\theta}{ds} \;=\; -\theta^2 + 2\mu \quad \forall s \in \mathbb{R}. \tag{24}$$

The complete (non-singular) solutions to (24) are the constant solutions $\theta(s) = \pm\sqrt{2\mu}$ and (up to a constant shift of s)

$$\theta(s) \;=\; \sqrt{2\mu}\tanh(\sqrt{2\mu}\,s). \tag{25}$$

The second differential Bianchi identity (11) implies that $g(s) := (\mu + f)\circ\gamma$ satisfies $g' = -\theta\,g$, and inserting (25) yields the general solution

$$g(s) \;=\; g_0/\cosh(\sqrt{2\mu}s)\,,$$

which contradicts $g > \mu$. Similarly, the constant solutions $\theta(s) = \pm\sqrt{2\mu}$ yield $g(s) = g_0\,e^{\mp\sqrt{2\mu}\,s}$, and hence also contradict $g > \mu$. Thus H vanishes nowhere on M. Next, consider the function $\ell(s) := ((1/H)\circ\gamma)(s)$; then (23) and the first equation in (22) together yield

$$\frac{d^2\ell}{ds^2} \;=\; 2 + 2\mu\,\ell \quad \forall s \in \mathbb{R},$$

whose general solution is $\ell(s) = -1/\mu + c_1\,e^{\sqrt{2\mu}\,s} + c_2\,e^{-\sqrt{2\mu}\,s}$. This means that either ℓ is constant along γ, or diverges as s goes to (at least one of) $\pm\infty$. Consider the latter case. It is straightforward to show that $\ell(s)$ satisfies $\ell' = \theta\,\ell$, which together with $g' = -\theta g$ implies that $g = c/\ell$ for some constant c. But this contradicts $g > \mu$, so we conclude that H must be a nonzero constant along all integral curves of $\boldsymbol{k}$. Then by (23) $\operatorname{div}\boldsymbol{k} = 0$, which implies that $\boldsymbol{k}[f] = 0$; but the first part of the proof showed that this is impossible. $\square$

4. The signature $(0, +, -)$

Proof of Theorem 2. Consider first when (M, g) is complete with globally constant principal Ricci curvatures $0, \lambda, -\lambda$; we then assume the existence

of a unit length vector field $\boldsymbol{k} \in \mathfrak{X}(M)$ with geodesic flow ($\nabla_{\boldsymbol{k}}\boldsymbol{k} = 0$) and satisfying $\widehat{\text{Ric}}(\boldsymbol{k}) = 0$. Let $\{\boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}\}$ be a local orthonormal basis of eigenvectors of $\widehat{\text{Ric}}$ such that

$$\widehat{\text{Ric}}(\boldsymbol{x}) \;=\; \lambda\boldsymbol{x} \quad , \quad \widehat{\text{Ric}}(\boldsymbol{y}) \;=\; -\lambda\boldsymbol{y}.$$

If $\{\boldsymbol{k}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ is the corresponding local complex tetrad, then $\text{Ric}(\boldsymbol{m}, \boldsymbol{m}) = \lambda$ and all other components are zero (with respect to $\{\boldsymbol{k}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$). Accordingly, the differential Bianchi identities (10) and (11) yield, respectively,

$$\beta = 0 \quad , \quad (\sigma + \bar{\sigma})\lambda \;=\; 0, \tag{26}$$

while the curvature identities (5) and (6) reduce to

$$\boldsymbol{k}[\rho] \;=\; |\sigma|^2 + \rho^2, \tag{27}$$

$$\boldsymbol{k}[\sigma] \;=\; 2\sigma\varepsilon + \sigma(\rho + \bar{\rho}) + \lambda. \tag{28}$$

Since $\beta = 0$ and σ is imaginary, an additional equation is provided via (9), which reads

$$|\sigma|^2 - |\rho|^2 - i\,\omega\varepsilon \;=\; 0. \tag{29}$$

Now suppose that ω is zero at a point p in the domain of $\{\boldsymbol{k}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$; since the imaginary part of (27) yields $\boldsymbol{k}[\omega] = -(\text{div}\,\boldsymbol{k})\,\omega$ as usual, ω must vanish along the (complete) integral curve $\gamma(s)$ of $\boldsymbol{k}$ through p. Then by (29), $|\sigma|^2 = |\rho|^2 = (1/4)(\text{div}\,\boldsymbol{k})^2$ everywhere along γ (recall that both $|\sigma|^2$ and $|\rho|^2$ are frame-independent), so that the real part of (27) gives

$$\frac{d\theta}{ds} \;=\; -\theta^2, \tag{30}$$

where, as before, we have set $\theta(s) = (\text{div}\,\boldsymbol{k} \circ \gamma)(s)$; because $\boldsymbol{k}$ has complete flow, θ is defined for all $s \in \mathbb{R}$. Then (30) implies that θ is either identically zero or else strictly positive, but in fact the second case cannot occur: if $\theta > 0$, then $1/\theta(s) = s + b$, a contradiction. On the other hand, if $\theta(s) \equiv 0$, then $\lambda = 0$ by (28), a contradiction once again. Thus we conclude that in fact ω must be nowhere vanishing on M; i.e., that $\{p \in M : \omega(p) \neq 0\} = M$. (We mention in passing that this is equivalent to the 1-form $g(\boldsymbol{k}, \cdot)$ being a contact form on M; see below.) Next, observe that

$$\boldsymbol{k}[|\sigma|^2] \;=\; -2(\text{div}\,\boldsymbol{k})|\sigma|^2 \quad , \quad \boldsymbol{k}[|\rho|^2] \;=\; -(\text{div}\,\boldsymbol{k})(|\sigma|^2 + |\rho|^2), \tag{31}$$

the former via (26), (28) and $\varepsilon + \bar{\varepsilon} = 0$, the latter via (27) (recall that $|\sigma|^2$ and $|\rho|^2$ are frame-independent; also, note that these equations can be defined everywhere along a given (geodesic) integral curve $\gamma(s)$ of $\boldsymbol{k}$, by parallel translating two orthonormal vectors $x, y \in \boldsymbol{k}^{\perp}_{\gamma(0)}$ along γ and writing down (31) with respect to their parallel translates). Armed with these, as well as with $\boldsymbol{k}[\omega] = -(\text{div}\,\boldsymbol{k})\,\omega$, the derivative of (29) along $\boldsymbol{k}$ simplifies to give $-i\,\omega\boldsymbol{k}[\varepsilon] = 0$, hence that $\boldsymbol{k}[\varepsilon] = 0$ everywhere in the domain of $\{\boldsymbol{k}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$. In fact the spin coefficient $\varepsilon = i\langle\nabla_{\boldsymbol{k}}\boldsymbol{x}, \boldsymbol{y}\rangle$ is a constant here. This follows

from setting $\beta = \kappa = \mathrm{Ric}(\boldsymbol{k},\boldsymbol{m}) = 0$ in (8), to obtain $\boldsymbol{m}[\varepsilon] = 0$, and hence that ε is a constant: $\varepsilon = ic$. The significance of this fact is seen by taking the real part of (28) to obtain

$$-\,ic\,(\sigma - \bar{\sigma}) \;=\; \lambda \tag{32}$$

where we have used the fact that σ is imaginary. In other words, the shear σ is also a constant in the domain of $\{\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$; since $|\sigma|^2$ is frame-independent, it follows that $|\sigma|^2$ is a global constant on M. If this constant is zero, then by (28) we have $\lambda = 0$, a contradiction. Since it is nonzero, the first equation in (31) dictates that $\mathrm{div}\,\boldsymbol{k} = 0$, so that $\rho + \bar{\rho} = 0$. With this established, the real part of (27) now gives $|\sigma|^2 + \rho^2 \;=\; |\sigma|^2 - |\rho|^2 = 0$. But this combines with (29) to yield $\omega\,\varepsilon = 0$, hence that $\varepsilon = ic = 0$, hence that $\lambda = 0$ by (32), a contradiction once again. Thus λ cannot be constant.

Now suppose that M is closed and that $\boldsymbol{k}$, in addition to having geodesic flow, is also divergence-free, but relax the condition that the nonzero principal Ricci curvatures are constants. For clarity, let us write them now as $\pm f$, where $f > 0$ and locally a smooth function. Once again, observe that with respect to any local complex tetrad $\{\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$,

$$\kappa \;=\; \rho + \bar{\rho} \;=\; \mathrm{Ric}(\boldsymbol{k},\cdot) \;=\; \mathrm{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}}) \;=\; 0. \tag{33}$$

Set $\psi := \mathrm{Ric}(\boldsymbol{m},\boldsymbol{m})$; then ψ is nowhere vanishing (because f is so) and also the only nonzero component of the Ricci tensor with respect to $\{\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$. Inserting (33) into the second differential Bianchi identity (11), as well as into the curvature identities (5) and (6), yields, respectively,

$$\sigma\bar{\psi} + \bar{\sigma}\psi \;=\; 0,$$

$$\boldsymbol{k}[\rho] \;=\; |\sigma|^2 + \rho^2, \tag{34}$$

$$\boldsymbol{k}[\sigma] \;=\; 2\sigma\varepsilon + \psi. \tag{35}$$

Then just as with the first equation in (31), this time with $\mathrm{div}\,\boldsymbol{k} = 0$,

$$\boldsymbol{k}[|\sigma|^2] \;=\; 0. \tag{36}$$

Along a given integral curve $\gamma(s)$ of $\boldsymbol{k}$, set $h(s) := (|\sigma|^2 \circ \gamma)(s)$. If $h(0) = 0$, then by (36) h is everywhere zero, in which case $\psi \circ \gamma = 0$ by (35), contradicting the fact that it is nowhere vanishing. Thus $h > 0$ along γ and so $|\sigma|^2 > 0$ on M; because the real part of (34) is $|\sigma|^2 - \omega^2/4 = 0$, it follows that the global smooth function ω^2 is nowhere vanishing on M. Consider now the determinant H of the bundle endomorphism D given by (15) above; a computation shows that $H = \omega^2/4 - |\sigma|^2 + (\mathrm{div}\,\boldsymbol{k})^2/4 = 0$. Because ω is nowhere vanishing, it follows that D has nullity one at every point of M. Assume that M is simply connected; then the smooth line bundle $X := \ker D$ has a global unit section $\boldsymbol{x} \in \Gamma(X)$, which satisfies $\nabla_{\boldsymbol{x}}\boldsymbol{k} = 0$. Similarly, the orthogonal complement of X in $\boldsymbol{k}^{\perp}$, Y, is a smooth line bundle, so it, too,

has a global unit section $\boldsymbol{y} \in \Gamma(Y)$. Let $\{\boldsymbol{k}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ denote the complex tetrad associated to the global orthonormal frame $\{\boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}\}$. Then its spin coefficients ρ and σ in particular satisfy

$$\sigma \;=\; i\,\frac{\omega}{2} \;=\; \bar{\rho}$$

(this follows from (3), (4), and the fact that $\operatorname{div} \boldsymbol{k} = 0$). Note that ω is now a nowhere vanishing smooth function globally defined on M; we can, by considering $-\boldsymbol{x}$ if necessary, assume that $\omega > 0$. Next, using $\kappa = \operatorname{Ric}(\boldsymbol{k}, \boldsymbol{m}) = 0$, (7) simplifies to $\boldsymbol{x}[\omega] = -\sqrt{2}\,\omega\,\bar{\beta}$, whose real and imaginary parts are, respectively,

$$\boldsymbol{x}[\omega] \;=\; -(\operatorname{div} \boldsymbol{x})\,\omega \quad , \quad \langle \nabla_{\boldsymbol{x}} \boldsymbol{x}, \boldsymbol{y} \rangle\, \omega \;=\; 0. \tag{37}$$

Observe that because ω is nowhere vanishing, it follows from the second equation in (37) that $\nabla_{\boldsymbol{x}} \boldsymbol{x} = 0$, so that the flow of $\boldsymbol{x}$, like that of $\boldsymbol{k}$, is everywhere geodesic (furthermore, β is real). The imaginary part of (34) yields $\boldsymbol{k}[\omega] = 0$, hence $\boldsymbol{k}[\sigma] = (i/2)\,\boldsymbol{k}[\omega] = 0$, so that (35) yields $\psi = -i\omega\,\varepsilon$; it follows that $\psi = \operatorname{Ric}(\boldsymbol{m}, \boldsymbol{m})$ is real in the frame $\{\boldsymbol{k}, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$. Finally, consider (9). Substituting $\beta = \bar{\beta} = \frac{1}{\sqrt{2}} \operatorname{div} \boldsymbol{x}$, $\psi = -i\,\omega\,\varepsilon$, and $|\sigma|^2 = |\rho|^2$ into (9) yields simply

$$\boldsymbol{x}[\operatorname{div} \boldsymbol{x}] \;=\; -(\operatorname{div} \boldsymbol{x})^2 + \psi. \tag{38}$$

Let γ be an integral curve of $\boldsymbol{x}$ and set $\ell(s) := 1/(\omega \circ \gamma)(s) > 0$. Then (38) and the first equation in (37) combine to yield

$$\frac{d^2 \ell}{ds^2} \;=\; \psi(\gamma(s))\,\ell(s) \qquad \forall s \in \mathbb{R}. \tag{39}$$

This implies that $\psi \circ \gamma$ must be positive, for otherwise $\ell(s) > 0$ is incompatible with $\ell''(s) < 0$ (note that ψ can never be zero at any point, for then so would f). Since $\operatorname{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}}) = 0$ implies $\operatorname{Ric}(\boldsymbol{y}, \boldsymbol{y}) = -\operatorname{Ric}(\boldsymbol{x}, \boldsymbol{x})$, observe that $\operatorname{Ric}(\boldsymbol{x}, \boldsymbol{x}) = \operatorname{Ric}(\boldsymbol{m}, \boldsymbol{m}) = \psi > 0$ on M; since the latter is closed, it follows that $\operatorname{Ric}(\boldsymbol{x}, \boldsymbol{x}) \geq b$ for some positive constant b. The significance of this can be seen by considering the analogue of the curvature identity (5) *for* $\boldsymbol{x}$ (rather than $\boldsymbol{k}$). In other words, for the complex tetrad $\{\boldsymbol{x}, \boldsymbol{n}, \overline{\boldsymbol{n}}\}$, with $\boldsymbol{n} := \frac{1}{\sqrt{2}}(\boldsymbol{k} - i\boldsymbol{y})$ (and whose corresponding spin coefficients we distinguish with a superscript "$\sim$", noting that $\tilde{\kappa} = 0$ because $\boldsymbol{x}$ has geodesic flow), one obtains the pair of equations

$$\boldsymbol{x}[\operatorname{div} \boldsymbol{x}] \;=\; \frac{\tilde{\omega}^2}{2} - 2|\tilde{\sigma}|^2 - \frac{(\operatorname{div} \boldsymbol{x})^2}{2} - \operatorname{Ric}(\boldsymbol{x}, \boldsymbol{x}) \quad , \quad \boldsymbol{x}[\tilde{\omega}] \;=\; -(\operatorname{div} \boldsymbol{x})\,\tilde{\omega}.$$

But because $\operatorname{Ric}(\boldsymbol{x}, \boldsymbol{x}) \geq b$, these equations together imply that $\tilde{\omega}$ is nowhere vanishing on M, hence that $g(\boldsymbol{x}, \cdot)$ is a contact form, because

$$dg(\boldsymbol{x}, \cdot)(\boldsymbol{k}, \boldsymbol{y}) \;=\; -\tilde{\omega}.$$

Furthermore, since $\boldsymbol{x}$ has unit length and geodesic flow, it is the Reeb vector field of $g(\boldsymbol{x}, \cdot)$ (i.e., the unique vector field satisfying $g(\boldsymbol{x}, \boldsymbol{x}) = 1$ and

$\boldsymbol{x} \lrcorner\, dg(\boldsymbol{x},\cdot) = 0$). By the Weinstein conjecture in dimension three [12], it follows that $\boldsymbol{x}$ has a *closed* integral curve $\gamma(s)$. But on closed γ we cannot everywhere have $\ell''(s) > 0$, in contradiction with (39). This completes the proof when M is simply connected.

If M is not simply connected, then pass to the finite-sheeted cover $\pi\colon (\widetilde{F},\tilde{g}) \longrightarrow (M,g)$ trivializing the line bundles X and Y, which is compact with principal Ricci curvatures $0, f\circ\pi, -f\circ\pi$. Repeating our argument on $(\widetilde{F},\tilde{g})$, the proof is complete. $\square$

5. The signature $(0,+,+)$

Proof of Theorem 3. The principal Ricci curvatures of (M,g) are $0, f, f$, with f not assumed to be constant. For $\boldsymbol{k}$ as given and any local complex tetrad $\{\boldsymbol{k},\boldsymbol{m},\overline{\boldsymbol{m}}\}$, the Ricci tensor satisfies

$$\text{Ric}(\boldsymbol{k},\boldsymbol{k}) \;=\; \text{Ric}(\boldsymbol{k},\boldsymbol{m}) \;=\; \text{Ric}(\boldsymbol{m},\boldsymbol{m}) \;=\; 0 \quad , \quad \text{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}}) \;=\; \frac{S}{2}, \tag{40}$$

where S is the scalar curvature of (M,g) which, by assumption, is positive and satisfies $\boldsymbol{k}[S] = 0$. We now show that $\boldsymbol{k}$ is parallel: $\kappa = \rho = \sigma = 0$. Inserting (40) into the differential Bianchi identities (10) and (11) yields $\kappa = (\text{div}\,\boldsymbol{k}) = 0$ (we remark here in passing that these also follow from the contracted Bianchi identity). These in turn simplify the real part of (5) to

$$\frac{\omega^2}{4} - |\sigma|^2 \;=\; 0.$$

Now we proceed as in the proof of Thereom 2: D will have zero determinant everywhere, with matrix given by

$$\begin{bmatrix} 0 & \omega(p) \\ 0 & 0 \end{bmatrix}.$$

We will thus look to work in a frame satisfying

$$\sigma \;=\; i\,\frac{\omega}{2} \;=\; \bar{\rho}, \tag{41}$$

the difference here being that we do not know that ω is nowhere vanishing. Now, if $\omega = 0$, then $\rho = \sigma = 0$ and we are done. Our strategy is thus to show that the open subset

$$\mathscr{U} \;:=\; \{p \in M : \omega(p) \neq 0\}$$

is empty. Clearly it suffices to prove this for each connected component, so we may assume $\mathscr{U}$ is connected. We first consider the case where $\mathscr{U}$ is simply connected. The map D has constant rank 1 in $\mathscr{U}$, hence $X = \ker D|_{\mathscr{U}}$ and its orthogonal complement in $\boldsymbol{k}^{\perp}|_{\mathscr{U}}$, Y, are smooth real line bundles over $\mathscr{U}$. As $\mathscr{U}$ is simply connected they have smooth global sections $\boldsymbol{x} \in \Gamma(X)$ and $\boldsymbol{y} \in \Gamma(Y)$ of unit length, which together with $\boldsymbol{k}$ form an orthonormal frame on $T\mathscr{U}$ satisfying $\nabla_{\boldsymbol{x}}\boldsymbol{k} = 0$. With respect to this frame, the quantities ρ and

σ satisfy (41). Using $\kappa = \mathrm{Ric}(\boldsymbol{k}, \boldsymbol{m}) = 0$, (7) simplifies to $\boldsymbol{x}[\omega] = -\sqrt{2}\,\omega\,\bar{\beta}$, whose real and imaginary parts of are, respectively,

$$\boldsymbol{x}[\omega] \;=\; -(\mathrm{div}\,\boldsymbol{x})\,\omega \quad , \quad \langle \nabla_{\boldsymbol{x}}\boldsymbol{x}, \boldsymbol{y}\rangle\,\omega \;=\; 0. \tag{42}$$

Observe that because ω is nowhere vanishing in $\mathscr{U}$, the second equation in (42) gives $\nabla_{\boldsymbol{x}}\boldsymbol{x} = 0$, so that the flow of $\boldsymbol{x}$ is everywhere geodesic. The imaginary part of (5) yields a similar equation for $\boldsymbol{k}$, $\boldsymbol{k}[\omega] = -(\mathrm{div}\,\boldsymbol{k})\,\omega = 0$, and since $\boldsymbol{k}[\sigma] = (i/2)\,\boldsymbol{k}[\omega]$, (6) reduces to $\omega\,\varepsilon = 0$, hence $\varepsilon = 0$. Finally, substituting $\beta = \bar{\beta} = \frac{1}{\sqrt{2}}\mathrm{div}\,\boldsymbol{x}, \varepsilon = 0$, and $|\sigma|^2 = |\rho|^2$ into (9) yields simply

$$\boldsymbol{x}[\mathrm{div}\,\boldsymbol{x}] \;=\; -(\mathrm{div}\,\boldsymbol{x})^2 - \frac{S}{2}\cdot \tag{43}$$

(Compare (43) with (38) in the proof of Theorem 2 above.) Now let $\gamma\colon [0,b) \longrightarrow \mathscr{U}$ be an integral curve of $\boldsymbol{x}$ that is maximally extended to the right. We claim that $b = \infty$. Indeed, suppose b is finite. Because $\nabla_{\boldsymbol{x}}\boldsymbol{x} = 0$, γ is a geodesic, and thus right-extendible in M by completeness. It follows that for b finite the limit $\lim_{s\to b}\gamma(s)$ exists (in M), and is not in $\mathscr{U}$. Setting $\theta(s) := (\mathrm{div}\,\boldsymbol{x} \circ \gamma)(s)$ and $\theta_0 := \theta(0)$, the first equation in (42) yields

$$(\omega\circ\gamma)(s) \;=\; \omega_0\, e^{-\int_0^s \theta(u)\,du} \qquad \forall s \in [0,b), \tag{44}$$

where, without loss of generality, $\omega_0 := \omega(\gamma(0))$ can be chosen to be positive, by an appropriate choice of $\boldsymbol{y}$. By (43), $\theta(s)$ is strictly decreasing (recall that S is positive), so that $\theta(s) < \theta_0$ for all $s \in (0,b)$. Hence (44) satisfies

$$(\omega\circ\gamma)(s) \;>\; \omega_0\, e^{-\theta_0 s} \qquad \forall s \in (0,b). \tag{45}$$

As $\mathscr{U} = \{p \in M : \omega(p) \neq 0\}$ does not contain $\lim_{s\to b}\gamma(s)$, we conclude that $\lim_{s\to b}(\omega\circ\gamma)(s) = 0$. This contradicts (45); hence b must be infinite. Repeating the argument for $-\boldsymbol{x}$ implies that the flow of $\boldsymbol{x}$ is complete in $\mathscr{U}$. With that established, let γ be an integral curve of $\boldsymbol{x}$ and set $h(s) := 1/(\omega\circ\gamma)(s)$. Then (43) and the first equation in (42) combine to yield

$$\frac{d^2h}{ds^2} \;=\; -\frac{S(\gamma(s))}{2}h(s) \;<\; 0 \qquad \forall s \in \mathbb{R},$$

where the inequality is due to $S > 0$ and $h > 0$. Any positive function h on $\mathbb{R}$ with $h'' < 0$ lies below its tangent lines, so that it must either cross zero at some point, or be everywhere constant, which is impossible as $h'' < 0$. The only way to avoid this contradiction is for $\mathscr{U}$ to be empty.

We now return to the general case when $\mathscr{U}$ is connected but not simply connected. Lifting to the Riemannian universal covering space $\pi\colon \widetilde{\mathscr{U}} \longrightarrow \mathscr{U}$, the lifts $\widetilde{X}$ and $\widetilde{Y}$ of X and Y have global unit length sections $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$, and $\boldsymbol{k}$ lifts to a unit length vector field $\tilde{\boldsymbol{k}}$. The argument proceeds as before: any integral curve of $\tilde{\boldsymbol{x}}$ is a geodesic $\tilde{\gamma}$ in $\widetilde{\mathscr{U}}$, which projects to a geodesic $\gamma = \pi\circ\tilde{\gamma}$ in $\mathscr{U}$. Completeness of M applied to γ once again implies that the flow of $\tilde{\boldsymbol{x}}$ is complete in $\widetilde{\mathscr{U}}$. The argument of the previous paragraph

then gives a contradiction unless $\widetilde{\mathscr{U}}$ is empty. This completes the proof that $\boldsymbol{k}$ is parallel. Finally, observe that the universal cover of (M, g) splits isometrically as $\mathbb{R}\times\widetilde{N}$; this follows from the de Rham decomposition theorem (see, e.g., [8, Theorem 56, p. 253] and also [2]). Note that when the scalar curvature is constant, then $\widetilde{N} = S^2$. □

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